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A SAMPLING THEOREM ON HOMOGENEOUS MANIFOLDS

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Abstract. We consider a generalization of entire functions of spherical exponential type and Lagrangian splines on manifolds. An analog of the Paley-Wiener theorem is given. We also show that every spectral entire function on a manifold is uniquely determined by its values on some discrete sets of points.

The main result of the paper is a formula for reconstruction of spectral entire functions from their values on discrete sets using Lagrangian splines.

1. Introduction and main results

The classical Shannon-Whittaker sampling theorem says, that if $f \in L^2(R)$ and its Fourier transform $\hat{f}$ has support in $[-\omega, \omega]$, then f is completely determined by its values at points $n\Omega$, where $\Omega = \pi/\omega$ and in the L^2-sense

$$f(t) = \sum f(n\Omega)\frac{\sin(\pi(t-n\Omega))}{\pi(t-n\Omega)}.$$

Functions $f \in L^2(R)$ with property $\mathrm{supp}\hat{f} \subset [-\omega, \omega]$ form a Paley-Wiener class PW_ω. The Paley-Wiener theorem states that f is in PW_ω if and only if f is an entire function of exponential type ω.

Entire functions of finite exponential type are also uniquely determined and can be recovered from their values on specific irregular sets of points x_n. As was shown by Paley and Wiener it is enough to assume that functions $\exp ix_nt, n \in Z$, form a Riesz basis for $L^2([-\pi, \pi])$.

One can consider even more general assumptions about the sequence x_n. The new and old results in the case when functions $\exp ix_nt$ form different kinds of frames in $L^2([-\omega, \omega])$ were summarized by J. Benedetto [1].

On the other hand, I. Schoenberg [11] used cardinal splines for reconstruction formula for the sequence $x_n = n$. This result was recently generalized by Lyubarskii and Madych [6] in the case when functions $\exp ix_nt, n \in Z$, form a Riesz basis for $L^2([-\pi, \pi])$.

In the compact case on the circle the similar statement takes place: every polynomial of degree n is completely defined by any $2n+1$ points and can be reconstructed using Lagrange polynomials. This reconstruction formula is perfect, except that Lagrange polynomials tend to oscillate for large number of knots. Even in this case the reconstruction by splines has an advantage (for example, for numerical integration) because splines have minimal curvature.

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We consider a generalization of Lagrangian splines on homogeneous manifolds and introduce an appropriate generalization of entire functions of spherical exponential type which we call spectral entire functions of exponential type. Our goal is to show that even on manifolds the reconstruction of irregularly sampled spectral entire functions of exponential type by splines is possible as long as the distance between points from a sampling sequence $x_n \in M$ is small enough. The known proof of the Shannon-Whittaker formula uses the fact that functions e^{int} form orthonormal basis in $L_2([-\pi,\pi])$. Our explanation of this phenomenon is different: an entire function of exponential type can be reconstructed from its values on certain discrete sets because it satisfies the Bernstein inequality. A subelliptic version of this result was considered by the author in [8] and [9].

All our results take place for a self-adjoint operator with C^∞-bounded coefficients on a manifold with bounded geometry [3], [10]. To reduce the amount of necessary definitions we consider the Laplace-Beltrami operator on a homogeneous manifold.

The following is a brief description of our main results.

Let M be a C^∞-homogeneous manifold, i.e. the group of isometries is transitive on M. The Laplace-Beltrami operator Δ of the corresponding Riemmanian metric $dist(x,y),\ x,y,\in M$, and the operator $D=\Delta^{1/2}$ are self-adjoint positive definite operators in the corresponding Hilbert space $L^2(M)$. According to the spectral theory [4] there exist a direct integral of Hilbert spaces $A=\int A(\lambda)dm(\lambda)$ and a unitary operator F from $L_2(M)$ onto A, which transforms the domain of D^k onto $A_k=\{a\in A|\lambda^k a\in A\}$ with norm

$$\|a(\lambda)\|_{A_k}=\left(\int_0^\infty \lambda^{2k}\|a(\lambda)\|^2_{A(\lambda)}dm(\lambda)\right)^{1/2}$$

besides $F(D^k f)=\lambda^k(Ff)$, if f belongs to the domain of D^k. As is known, A is the set of all m-measurable functions $\lambda\to a(\lambda)\in A(\lambda)$, for which the norm

$$\|a\|_A=\left(\int_0^\infty \|a(\lambda)\|^2_{A(\lambda)}dm(\lambda)\right)^{1/2}$$

is finite.

We will say that a function f from $L_2(M)$ is a spectral entire function of exponential type ω (ω-SE function) if its "Fourier transform" Ff has support in $[0,\omega]$. The $E_\omega(D)$ will denote the set of all ω-SE functions. The next theorem from section 5 can be considered as an abstract version of the Paley-Wiener theorem. Throughout the paper norm $\|f\|$ means $L_2(M)$-norm.

Theorem 1.1. The following conditions are equivalent:

a) function f belongs to $E_\omega(D)$;

b) function f satisfies Bernstein inequality

$$\|D^k f\|\le\omega^k\|f\|$$

for every natural k;

c) for any $h\in L_2(M)$ the complex valued function of one variable $t\in(-\infty,\infty)$

$$F(t)=\langle e^{itD}f,h\rangle=\int_M e^{itD}f\overline{h}d\mu$$

is an entire function of exponential type ω which is bounded on the real line, i.e. it has analytic extension to the complex plane C and there exists a constant $a=a(h)$

such that

$$|F(z)| \leq ae^{\omega|z|}, \qquad z \in C;$$

d) the abstract function $e^{itD}f$ has continuation to the plane as an entire function and there exists a constant b such that

$$\|e^{izD}f\| \leq be^{\omega|z|}, \qquad z \in C.$$

Here are some examples.

1. If $M = R^d$ and Δ is the usual Laplace operator then we can show that $E_\omega(D)$ is the set of all entire functions of spherical exponential type ω.

2. If M is compact, then $\bigcup_{\omega>0} E_\omega(D) = E(D)$ is the linear span of all eigenfunctions of D. In the case when Δ is a Laplace-Beltrami operator of invariant metric and M is equivariantly imbedded in R^q then, $E(D)$ can be identified with restrictions of polynomials in R^q onto M, [3], [7].

3. In the case of the non-compact symmetric space and the corresponding Laplace-Beltrami operator the space $E(D)$ consists of all L^2 functions whose Fourier-Helgason transform has compact support [3], [6].

The $Y(r, \lambda)$ will denote the set of all sets of points $Z = \{x_\gamma\}$ such that

a) $\inf_{\gamma \neq \mu} dist(x_\gamma, x_\mu) > 0$;

b) balls $B(x_\gamma, \lambda)$, form a cover of M with multiplicity $\leq r$.

The latter means that every ball has non-empty intersections with no more than r other balls from this family. The discussion of existence and construction such sets of points is given in the second section.

Lemma 1.2. There exist integers $r = r(M) > 0$ and $\rho(M) > 0$ such that the set $Y(r(M), \lambda)$ is not empty as long as $\lambda \leq \rho(M)$.

In section 2 we also discuss the analysis on manifolds. We need a notion of an elliptic operator and corresponding regularity theory in the Sobolev scale of spaces $H^s(M), -\infty < s < \infty$.

Let $s > d/2, \lambda \leq \rho(M)$, and $Z \in Y(r(M), \lambda)$. We introduce $U^s(Z)$, the space of all functions in $H^s(M)$ whose restriction on Z is zero.

The main goal of section 3 is to prove the following theorem.

Theorem 1.3. There exist $\lambda_0 \geq 0, C_0 \geq 0$ such that for every $\lambda \leq \lambda_0$ every $Z \in Y(r(M), \lambda)$ and every $f \in U^k(Z), k = 2^l d, d = dimM, l = 1, 2, ...,$ the following inequality takes place

$$\|f\| \leq (C_0\lambda)^k \|D^k f\|,$$

where $D = \Delta^{1/2}$.

In section 4 we construct Lagrangian splines. A similar approach was used by Lemarie [4].

Theorem 1.4. For any function f from $H^{2k}(M), k = 2^l d, l = 1, 2, ...,$ there exists a unique function $s_k(f)$ from the Sobolev space $H^{2k}(M)$, such that

a) $f|_Z = s_k(f)|_Z$;

b) $s_k(f)$ minimizes functional $u \to \|\Delta^k u\|$.

Every such function $s_k(f)$ is of the form $\sum f(x_\gamma)L^{2k}_\gamma$ where the function $L^{2k}_\gamma \in H^{2k}(M), x_\gamma \in Z$ minimizes the same functional and takes value 1 at the point x_γ and 0 at all other points of Z. These functions L^{2k}_γ form a Riesz basis in the space of all polyharmonic functions with singularities on Z, i.e. in the space of such functions from $H^{2k}(M)$ which in the sense of distributions satisfy equation

$$\Delta^{2k}u = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma)$$

where $\delta(x_\gamma)$ is the Dirac measure at the point x_γ. If in addition the set Z is invariant under some subgroup of diffeomorphisms acting on M, then every two functions $L^{2k}_\gamma, L^{2k}_\mu$ are translates of each other.

In the case of euclidean space such functions L^{2k}_γ are called Lagrangian splines. We have the following direct approximation theorem.

Theorem 1.5. There exists $\lambda_0 > 0, c_0 > 0$ which depend only on the manifold and operator Δ such that for the given $Z \in Y(r(M), \lambda)$ with $\lambda < \lambda_0$ the following estimate takes place

$$\|f - s_k(f)\| \le (c_0\lambda)^k \|\Delta^{k/2} f\|, \qquad f \in H^k(M), k = 2^l d, l = 1, 2, ...$$

The main result of the paper is the following theorem.

Theorem 1.6. For the same constants $\lambda_0 > 0$ and $c_0 > 0$ as above

a) every ω-SE function $f \in E_\omega(D), \omega > 0$ is uniquely determined by its values on any set $Z \in Y(r(M), \lambda)$ as long as $\lambda < (c_0\omega)^{-1}$;

b) for every such set Z the sequence of splines $s_k(f) = \sum f(x_\gamma) L^{2k}_\gamma, k = 2^l d, l = 1, 2, ...,$ converges to $f \in E_\omega(D)$ in $L^2(M)$-norm.

The quantity $(c_0\omega)^{-1}$ is an analog of the Nyguist sampling rate in the classical case.

2. Analysis on homogeneous manifolds

Let M, $\dim M = d$ be a C^∞-homogeneous Riemannian manifold. The $B(x, \rho)$ will denote a ball whose center is $x \in M$ and radius is $\rho > 0$. The measure of this ball $B(x, \rho)$ is independent of x and will be denoted by $v(\rho)$. The notation $Y(r, \lambda)$ was introduced in the Introduction. It is clear that $Y(r_1, \lambda_1) \subset Y(r_2, \lambda_2)$ if $r_1 \ge r_2$ and $\lambda_1 \le \lambda_2$.

Denote by $T_x(M)$ the tangent space of M at a point $x \in M$ and let $exp_x : T_x(M) \to M$ be the exponential geodesic map, i.e. $exp_x(u) = \gamma(1), u \in T_x(M)$ where $\gamma(t)$ is the geodesic starting at x with the initial vector $u : \gamma(0) = x, \frac{d\gamma(0)}{dt} = u$. If the $inj(M)$ is the injectivity radius of M then the exponential map is a diffeomorphism of a ball of radius $\rho < inj(M)$ in the tangent space $T_x(M)$ onto the ball $B(x, \rho)$. Using L'Hôpital rule one can show

$$\lim_{\rho \to 0} \frac{v(2\rho)}{v(\rho)} = 2^d,$$

where $d = \dim M$. It implies the doubling property of Riemannian metric: there exists a constant k which depends on Riemannian structure such that

$$v(2\rho) \le k v(\rho), \rho \le inj(M).$$

Fix a point $x \in M$ and consider ball $B(x, \lambda/4), \lambda \le inj(M)$. Using isometries we can construct a family of disjoint balls $B(x_i, \lambda/4)$ such that there is no ball $B(x, \lambda/4), x \in M$, which has non-empty intersections with balls from our family. The same property implies that the family $B(x_i, \lambda/2)$ is a cover for M. Now, if $x \in B(x_i, \lambda)$ then $B(x_i, \lambda/4) \subset B(x, 2\lambda)$. Since any two balls from the family $B(x_i, \lambda/4)$ are disjoint, it gives the estimate for the index of multiplicity r of the cover $B(x_i, \lambda)$ i.e. $r \leq v(2\lambda)/v(\lambda/4)$ and using doubling property we obtain the estimate

$$r \leq \frac{v(2\lambda)}{v(\lambda/4)} \leq r(M).$$

So, we proved Lemma 1.2 from Introduction.

Using any cover $\{B(x_i, \lambda)\}$ of finite multiplicity one can construct the partition of unity with following properties.

Lemma 2.1. For every $\lambda \leq inj(M)$ there exists a non-negative partition of unity $\{\psi_i\} \in C_0^\infty(M)$ such that

a) $\operatorname{supp} \psi_i \subset B(x_i, \lambda)$,

b) $|\psi_i^{(\alpha)}(x)| \leq C(\alpha), C(\alpha)$ is independent on i for every multi-index α in the coordinate system defined by *exp*.

By means of such partitions of unity it is possible to construct the Sobolev spaces $H^s(M), -\infty < s < \infty$. On $C_0^\infty(M)$ the following norm is considered

$$\|g\|_s^2 = \sum_{i=1}^{\infty} \|\psi_i g\|_s^2$$

where from the right we have the usual Sobolev norm. Then the Sobolev space $H^s(M)$ is introduced as the completion of $C_0^\infty(M)$ with respect to this norm [10]. The spaces $H^s(M)$ have the same properties as the standard Sobolev spaces on R^n. In particular the scalar product on $C_0^\infty(M)$ can be extended to a continuous pairing $\langle .,. \rangle$ between $H^s(M)$ and $H^{-s}(M)$ and the latter are dual to each other with respect to it. The following facts about the Laplace-Beltrami operator Δ can be found in [3], [10], [12], [13]. The closure from $C_0^\infty(M)$ of Δ in $L^2(M)$ is self-adjoint and the positive definite operator and domain of $D^s, s > 0, D = \Delta^{1/2}$ is exactly Sobolev space $H^s(M)$. Using duality between $H^s(M)$ and $H^{-s}(M)$, the operator $\Delta^k, k > 0$, from the space $H^s(M)$ can be extended to the space $H^{-s}(M)$ and we have

$$\langle \Delta^k f, h \rangle = \langle f, \Delta^k h \rangle,$$

for $f \in H^s(M), h \in H^{-s-2k}(M), -\infty < s < \infty$.

The operator Δ has the form $\sum_{|\alpha| \leq 2} a_\alpha(x)\partial^\alpha$ in any geodesic coordinate system in the neighborhood $B(x_i, \rho)$ and since it is an invariant operator the estimate $|\partial^\gamma a_\alpha| \leq C_\gamma$ takes place uniformly with respect to i. The invariance of this operator also implies uniform ellipticity in the sense that there exists a constant $C > 0$ such that uniformly to $(x, \xi) \in M \times T^*(M)$

$$|a(x, \xi)| \geq C|\xi|^2, \qquad (x, \xi) \in M \times T^*(M).$$

Here $a(x, \xi)$ is the principal symbol of Δ and $|\xi|$ is the distance on a cotangent space $T^*(M)$ with respect to the given Riemannian structure on M.

For such a C^∞-bounded uniformly elliptic differential operator on the manifold with bounded geometry M the following regularity properties are simple consequences of the corresponding results in R^d and can be proved using partition of unity from Lemma 2.1 (see [2], [12], [13]).

Lemma 2.2. a) If $k > 0$ is an integer, then on $H^k(M)$ norms $\|(I+\Delta)^{k/2}f\|, \|f\| + \|\Delta^{k/2}f\|$ and $\|f\|_{H^k(M)}$ are equivalent.

b) The map $\Delta : H^s(M) \to H^{s-2}(M), s \in R$ is continuous.

c) For any $s, t \in R$ there exists $c > 0$ such that

$$\|g\|_s \le c(\|\Delta g\|_{s-2} + \|g\|_t), \qquad g \in C_0^\infty(M);$$

d) If $g \in \bigcup_{s\in R} H^s(M)$ and $\Delta g \in H^{s-2}(M), s \in R$, then $g \in H^s(M)$.

3. Basic inequalites

The first lemma is a part of mathematical folklore.

Lemma 3.1. If S generates a C_0 one-parameter group of operators e^{tS} such that $\|e^{tS}f\| = \|f\|$, then for every $n \ge 2$ there exists a $C(n)$ such that for all $\varepsilon > 0$ all $1 \le m \le n-1$ and all f in the domain of S^n

$$\|S^m f\| \le \varepsilon^{n-m}\|S^n f\| + \varepsilon^{-m}C(n)\|f\|.$$

Proof. According to the Hille-Phillips-Yosida theorem the assumptions imply

$$\|(I + \varepsilon S)^{-1}\| \le 1$$

and the same for the operator $(I - \varepsilon S)$. Then

$$\|f\| \le \|(I + \varepsilon S)f\|$$

and the same for the operator $(I - \varepsilon S)$. It gives

$$\varepsilon\|Sf\| \le \|(I - \varepsilon S)f\| + \|f\| \le \|(I + \varepsilon^2 S^2)f\| + \|f\| \le \varepsilon^2\|S^2 f\| + 2\|f\|.$$

So, for any f from the domain of S^2 we have inequality

$$\|Sf\| \le \varepsilon\|S^2 f\| + 2/\varepsilon\|f\|, \qquad \varepsilon > 0.$$

The general case can be proved by induction. □

Lemma 3.2. For the same operator S as above if for some f from the domain of S

$$\|f\| \le A + a\|Sf\|, \qquad a > 0,$$

then for all $m = 2^l, l = 0, 1, 2, ...,$

$$\|f\| \le mA + 8^{m-1}a^m\|S^m f\|$$

as long as f belongs to the domain of S^m.

Proof. As was shown above

$$\|Sf\| \le \varepsilon\|S^2 f\| + 2/\varepsilon\|f\|, \qquad \varepsilon > 0.$$

Now, our inequality is true for $m = 1$. If it is true for m then

$$\|f\| \le mA + 8^{m-1}a^m(\varepsilon\|S^{2m}f\| + 2/\varepsilon\|f\|).$$

Setting $\varepsilon = 8^{m-1}(a)^m 2^2$, we obtain

$$\|f\| \le 2mA + 8^{2m-1}(a)^{2m}\|S^{2m}f\|.$$

Lemma 3.2 is proved. □

Lemma 3.3. There exist $\lambda_0 = \lambda(M)$ and $C = C(M)$ such that for any $f \in C_0^\infty(M)$ and any $0 < \lambda < \lambda_0$

$$\|f\| \leq C\left(\lambda^{d/2}(\sum_\gamma |f(x_\gamma)|^2)^{1/2} + \lambda^d\|\Delta^{d/2}f\|\right).$$

Proof. We assume that λ is smaller than the injectivity radius; then $B(x_\gamma, \lambda)$ belongs to a geodesic coordinate system. The Taylor formula gives

$$f(x) = f(x_\gamma) + \sum_{1\leq|\alpha|<k} C_\alpha \partial^\alpha f(x)(x - x_\gamma)^\alpha$$

$$+ \sum_{|\alpha|=k} C_\alpha \int_0^\rho t^{k-1}\partial^\alpha f(x_\gamma + t\theta)\theta^\alpha dt$$

where $\alpha = (\alpha_1, ..., \alpha_d), (x - x_\gamma)^\alpha = (x_1 - x_{1,\gamma})^{\alpha_1}...(x_d - x_{d,\gamma})^{\alpha_d}; \theta^\alpha = \theta_1^{\alpha_1}...\theta_d^{\alpha_d}; \theta = (x - x_\gamma)/\rho, \rho = |x - x_\gamma|$ and ∂^α is a mixed partial derivative.

It is evident that the first sum is dominated by

$$C\sum_{j=1}^{k-1}\lambda^j \sum_{1\leq|\alpha|\leq j} \|\partial^\alpha f\|_{L^2(B(x_\gamma,\lambda))}$$

for some $C = C(k) \geq 0$. Next, using Schwartz inequality and the assumption that $k > d/2$ we obtain

$$|\int_0^\rho \partial^\alpha f(x_\gamma + t\theta)\theta^\alpha dt|^2 \leq C\rho^{2k-d}\int_0^\rho |\partial^\alpha f(x_\gamma + t\theta)|^2 t^{d-1}dt.$$

We integrate both sides of this inequality in the spherical coordinate system (ρ, θ). Changing the order of integration in t and ρ we obtain that the $L_2(B(x_\gamma, \lambda))$-norm of the term

$$\int_0^\rho t^{k-1}\partial^\alpha f(x_\gamma + t\theta)\theta^\alpha dt$$

is dominated by

$$\|f\|_{L_2(B(x_\gamma,\lambda))} \leq \lambda^{d/2}|f(x_\gamma)| + \sum_{j=1}^{k}\lambda^j \sum_{i\leq|\alpha|\leq j}\|\partial^\alpha f\|_{L_2(B(x_\gamma,\lambda))}.$$

Suppose that the set of points $Z = x_\gamma$ belongs to $Y(r(M), \lambda), \lambda$ is smaller than the injectivity radius. Summation over all balls in corresponding cover gives

$$\|f\| \leq C\left\{\lambda^{d/2}(\sum |f(x_\gamma)|^2)^{1/2} + \sum_{j=i}^{k}\lambda^j\|f\|_{H_j(M)}\right\}$$

$$\leq C\left\{\lambda^{d/2}(\sum_\gamma |f(x_\gamma)|^2)^{1/2} + \sum_{j=1}^{k}\lambda^j(\|f\| + \|\Delta^{j/2}f\|)\right\}.$$

Repeated applications of Lemma 3.1 with $\varepsilon = a\lambda$ lead to the inequality

$$\lambda^j\|\Delta^{j/2}f\| \leq a^{d-j}\lambda^d\|\Delta^{d/2}f\| + C(d)a^{-j}\|f\|.$$

If a is large enough, we come to the inequality

$$\|f\| \le C(d)\left\{\lambda^{d/2}(\sum_{\gamma}|f(x_\gamma)|^2)^{1/2}+\lambda\|f\|+\lambda^d\|\Delta^{d/2}f\|\right\}$$

and for small λ it gives

$$\|f\| \le C(d)\left\{\lambda^{d/2}(\sum_{\gamma}|f(x_\gamma)|^2)^{1/2}+\lambda^d\|\Delta^{d/2}f\|\right\}.$$

Lemma 3.3 is proved. □

Now to prove Theorem 1.3 we use Lemma 3.2 which gives us

$$\|f\|_{L^2(M)} \le C\left(m\lambda^{d/2}(\sum|f(x_\gamma)|^2)^{1/2}+8^{m-1}(\lambda^d)^m\|(\Delta^{d/2})^m f\|\right),$$

where $m=2^l, l=0,1,2,....$ In particular, if $f\in U_k(Z), k=2^l d, l=0,1,...,$

$$\|f\|_{L^2(M)} \le (C\lambda)^k\|\Delta^{k/2}f\|.$$

Theorem 1.3 is proved.

Lemma 3.4. 1) For any $k>d/2$

$$\|\Delta^{k/2}f\|+\left(\sum_{\gamma}|f(x_\gamma)|^2\right)^{1/2} \le C_k\|f\|_{H^k(M)}.$$

2) If $k=2^l d, d=$dim M, $l=0,1,2,...,Z\in Y(r(M),\lambda),\lambda<\lambda_0$, then the above two norms are equivalent.

Proof. In order to prove the inequality we consider the $C_0^\infty(M)$ functions φ_γ with disjoint supports such that $\varphi_\gamma(x_\gamma)=1$. Using the Sobolev embedding theorem we obtain for any natural $k>d/2$

$$\left(\sum_{\gamma}|f(x_\gamma)|^2\right)^{1/2} \le C_k\left(\sum_{\gamma}\|f\varphi_\gamma\|^2_{H^k(M)}\right)^{1/2} \le C_k\|f\|_{H^k(M)}.$$

To prove the second part of the lemma observe that we have already proved the inequality

$$\|f\| \le C_k\left\{\left(\sum_{\gamma}|f(x_\gamma)|^2\right)^{1/2}+\|\Delta^{k/2}f\|\right\}, \qquad k=2^l d.$$

This inequality implies for $k=2^l d$

$$\|f\|_{H^k(M)} \le C_k\left\{\|f\|+\|\Delta^{k/2}f\|\right\} \le C_k\left\{\left(\sum_{\gamma}|f(x_\gamma)|^2\right)^{1/2}+\|\Delta^{k/2}f\|\right\}.$$

The proof of Lemma 3.4 is over. □

4. Splines on manifolds

Given a $Z \in Y(\lambda, r)$ and a sequence $\{s_\gamma\} \in l_2$ we will be interested to find a function $s_k \in H^{2k}(M), k$ large enough, such that

a) $s_k(x_\gamma) = s_\gamma, x_\gamma \in Z$;

b) function s_k minimizes functional $u \to \|\Delta^k u\|$.

The same problem for functional $u \to \|u\|_{H^{2k}(M)}, u \in H^{2k}(M), k > d/4$, can be solved easily.

For the given sequence $s_\gamma \in l_2$ consider a function f from $H^{2k}(M)$ such that $f(x_\gamma) = s_\gamma$. Let Pf denote the orthogonal projection of this function f (in the Hilbert space $H^{2k}(M)$ with natural inner product) on the subspace $U^{2k}(Z) = \left\{f \in H^{2k}(M) | f(x_\gamma) = 0\right\}$ with the $H^{2k}(M)$-norm. Then the function $g = f - Pf$ will be the unique solution of the above minimization problem for the functional $u \to \|u\|_{H^{2k}(M)}, k > d/4$.

The problem with functional $u \to \|\Delta^k u\|$ is that it is not a norm. But we already proved that for $k = 2^l d$ the norm $H^{2k}(M)$ is equivalent to the norm

$$\|\Delta^{2k} f\| + \left(\sum_{x_\gamma \in Z} |f(x_\gamma)|^2\right)^{1/2}.$$

So, the above procedure can be applied to the Hilbert space $H^{2k}(M)$ with the inner product

$$\langle f, g\rangle = \sum_{x_\gamma \in Z} f(x_\gamma) g(x_\gamma) + \langle \Delta^{k/2} f, \Delta^{k/2} g\rangle$$

and it clearly proves existence and uniqueness of the solution of our minimization problem for the functional $u \to \|\Delta^k u\|, k = 2^l d$.

The set of all $L_2(M)$-solutions of the equation

$$\Delta^{2k} u = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure and $\{\alpha_\gamma\} \in l_2$ will be denoted by $S^{2k}(M)$. Our next goal is to show that every $s_k(f) \in S^{2k}(M)$.

Indeed, suppose that $s_k \in H^{2k}(M)$ is a solution to the minimization problem and $h \in U^{2k}(Z)$. Then

$$\|\Delta^k(s_k + \lambda h)\|^2 = \|\Delta^k s_k\|_2^2 + 2Re\lambda \int_G \Delta^k s_k \Delta^k h d\mu + |\lambda|^2 \|\Delta^k h\|_2^2.$$

The function s_k can be a minimizer only if for any $h \in U^{2k}(Z)$

$$\int_M \Delta^k s_k \Delta^k h d\mu = 0.$$

So, the function $\Phi = \Delta^k s_k \in L_2(M)$ is orthogonal to $\Delta^k U^{2k}(Z)$. Let φ_γ be the same set of functions as above and $h \in C_0^\infty(M)$. Then the function $h - \sum h(x_\gamma)\varphi_\gamma$ belongs to the $U^{2k}(Z) \cap C_0^\infty(M)$. Thus,

$$0 = \int_M \Phi \overline{\Delta^k (h - \sum h_\gamma \varphi_\gamma)} d\mu = \int_M \Phi \overline{\Delta^k h} d\mu - \sum \overline{h(x_\gamma)} \int_M \Phi \overline{\Delta^k \varphi_\gamma} d\mu.$$

In other words

$$\Delta^k \Phi = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma),$$

or

$$\Delta^{2k} s_k = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure.

Moreover, for any integer $r > 0$

$$\sum_{\gamma=1}^{r} |\alpha_\gamma|^2 = \langle \sum_1^\infty \alpha_\gamma \delta(x_\gamma), \sum_1^r \alpha_\gamma \phi_\gamma \rangle \leq C \| \sum_1^\infty \alpha_\gamma \delta(x_\gamma) \|_{H^{-2k}(M)} (\sum_1^r |\alpha_\gamma|^2)^{1/2},$$

where C is independent on r. It shows that the sequence $\{\alpha_\gamma\}$ belongs to l_2.

Now suppose that $f \in H^\infty(M)$ and

$$\Delta^{2k} f = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma),$$

where $\{\alpha_\gamma\} \in l_2$.

Because of Lemma 3.4 for any $\varepsilon > 0$

$$|\langle \Delta^{2k} f, g \rangle| = |\langle \sum \alpha_\gamma \delta(x_\gamma), g \rangle|$$

$$\leq \left(\sum |\alpha_\gamma|^2 \right)^{1/2} \left(\sum |g(x_\gamma)^2 \right)^{1/2} \leq C \left(\sum |\alpha_\gamma|^2 \right)^{1/2} \|g\|_{H^{d/2+\varepsilon}(M)}.$$

It shows that the distribution $\sum_1^\infty \alpha_\gamma \delta(x_\gamma) = \Delta^{2k} f$ belongs to $H^{-d/2-\varepsilon}(M)$. Since the operator Δ^{2k} is C^∞-bounded and uniformly elliptic of order $4k$ we can use the corresponding regularity result from section 2, which gives that f belongs to $H^{-d/2-\varepsilon+4k}(M)$ which is included in $H^{2k}(M)$ for all $k > d$. The assertion that the orthogonal complement of $\Delta^k U^{2k}(Z)$ is a subset of $S^{2k}(Z)$ is proved. Conversely, if f, h belong to $S^{2k}(Z)$ and $U^{2k}(Z) \cap C^\infty$ respectively, then, since $f \in H^{2k}(M)$ and $h \in H^{2k}(M)$, then pairing $\langle ., . \rangle$ is an extension of the scalar product in $L^2(M)$,

$$\int_M f \overline{\Delta^k h} d\mu = \langle \Delta^k f, \overline{h} \rangle = \sum \alpha_\gamma \overline{h(x_\gamma)} = 0.$$

Thus we proved the following:

Lemma 4.1. A function $f \in L_2(M)$ satisfies equation

$$\Delta^{2k} f = \sum_{x_\gamma \in Z} \alpha_\gamma \delta(x_\gamma),$$

where $\{\alpha_\gamma\} \in l_2$ if and only if f is a solution to the minimization problem stated above.

Lemma 4.2. Every function from $S^{2k}(Z), k = 2^l d$, is uniquely determined by its values at points $x_\gamma \in Z$. In particular, for any $x_\gamma \in Z$ there exists a unique $L_\gamma^{2k}(Z) \in S^{2k}(Z)$ such that it takes value 1 at the point x_γ and 0 at all other points in Z. These functions form a Riesz basis in $S^{2k}(Z)$.

Recall that the last assertion means that for any $g \in S^{2k}(Z)$ in $L_2(M)$ we have $g = \sum_\gamma g(x_\gamma) L_\gamma^{2k}$ and there are $C_1, C_2 > 0$ such that

$$\|g\|_2 \leq C_1 \left(\sum |g(x_\gamma)|^2 \right)^{1/2} \leq C_2 \|g\|_2, \qquad g \in S^{2k}(Z).$$

Proof. Since $S^{2k}(Z)$ is closed in the $L_2(M)$-norm and $S^{2k}(Z) \subset H^{2k}(M)$, the $L_2(M)$-norm and $H^{2k}(M)$-norm are equivalent on $S^{2k}(Z)$. Moreover, one can show that on the space $S^{2k}(Z), k = 2^l d$, the norm $H^{2k}(M)$ is equivalent to the norm $(\sum |f(x_\gamma)|^2)^{1/2}, f \in S^{2k}(Z)$.

Indeed, if $\varphi_\gamma \in C^\infty$ have disjoint supports in $B(x_\gamma, \lambda/4)$ and $\varphi_\gamma(x_\mu) = \delta_{\gamma\mu}, |\varphi_\gamma| \leq 1$, then the function $F = \sum_{\gamma \in N} f(x_\gamma)\varphi_\gamma$ is in $H^{2k}(M)$ and $f(x_\gamma) = F(x_\gamma), k > d/2$. Because of the minimization property

$$\|\Delta^k f\| \leq \|\Delta^k F\| \leq C \left(\sum_\gamma |f(x_\gamma)|^2 \right)^{1/2}.$$

Since for $k = 2^l d$ the $H^{2k}(M)$ norm on S^{2k} is equivalent to the norm $\|\Delta^k f\|$ it implies equivalence of it to the norm $(\sum_\gamma |f(x_\gamma)|^2)^{1/2}$.

It shows that every $f \in S^{2k}(Z), k = 2^l d$, is completely determined by its values $f(x_\gamma)$. In particular for any $x_\gamma \in Z$ there exists a unique function $L_\gamma^{2k}(Z)$ in $S^{2k}(Z)$ such that it takes value 1 at the point x_γ and 0 at all other points in Z. Lemma 4.2 is proved. □

The last two lemmas give complete proof of Theorem 1.4 from the Introduction.

Next, if $f \in H^{2k}(M), k = 2^l d, l = 0, 1, \ldots$ and $Z \in Y(r(M), \lambda), \lambda < \lambda_0$, then $f - s_k(f) \in U^{2k}(Z)$ and by Theorem 1.3 we have for $k = 2^l d, l = 0, 1, \ldots,$

$$\|f - s_k(f)\| \leq (C_0\lambda)^k \|\Delta^{k/2}(f - s_k(f))\|.$$

Using the minimization property of $s_k(f)$ we obtain

$$\|f - s_k(f)\| \leq (c_0\lambda)^k \|\Delta^{k/2} f\|, \qquad k = 2^l d, l = 0, 1, \ldots.$$

Thus, we proved the approximation Theorem 1.5.

5. The spectral entire functions on manifolds and the sampling theorem

Let $M, \Delta, L_2(M)$ be as above. The goal of this section is to introduce an appropriate generalization of entire functions of exponential type. In a classical setting a function from $L_2(R)$ is an entire function of exponential type if its Fourier transform has compact support. In our situation Δ is a self-adjoint positive operator in the Hilbert space $L_2(M)$. We consider $D = \Delta^{1/2}$ and the domain of D^k is exactly Sobolev space $H^k(M)$. According to the spectral theory [5], there exist a direct integral of Hilbert spaces $A = \int A(\lambda)dm(\lambda)$ and a unitary operator F from $L_2(M)$ onto A, which transforms $H^k(M)$ onto $A_k = \{a \in A | \lambda^k a \in A\}$ with norm

$$\|a(\lambda)\|_{A_k} = \left(\int_0^\infty \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} dm(\lambda) \right)^{1/2}$$

besides $F(Df) = \lambda(Ff), f \in H^1(M)$.

We will say that the function f from $L_2(M)$ is a spectral entire function of exponential type ω or ω-SE function if its "Fourier transform" $Ff = a$ has support in $[0, \omega]$. The $E_\omega(D)$ will denote the set of all ω-SE functions.

The first theorem is evident.

Theorem 5.1. a) The set $\bigcup_{\omega>0} E_\omega(D) \subset C^\infty(M)$ is dense in $L_2(M)$;
b) the $E_\omega(D)$ is a linear closed subspace in $L^2(M)$.

We now prove that conditions a) and b) from Theorem 1.1 are equivalent.

Let f belong to the space $E_\omega(D)$ and $Ff = a \in A$. Then

$$\left(\int_0^\infty \lambda^{2k}\|a(\lambda)\|^2_{A(\lambda)}dm(\lambda)\right)^{1/2} = \left(\int_0^\omega \lambda^{2k}\|a(\lambda)\|^2_{A(\lambda)}dm(\lambda)\right)^{1/2} \le \omega^k\|a\|_A, \qquad k \in N,$$

which gives the Bernstein inequality for f.

Conversely, if f satisfies the Bernstein inequality, then $a = Ff$ satisfies $\|a\|_{A_k} \le \omega^k\|a\|_A$. Suppose that there exists a set $\sigma \subset [0,\infty] \setminus [0,\omega]$ whose m-measure is not zero and $a|_\sigma \neq 0$. We can assume that $\sigma \subset [\omega+\epsilon,\infty)$ for some $\epsilon > 0$. Then for any $k \in N$ we have

$$\int_\sigma \|a(\lambda)\|^2_{A(\lambda)}dm(\lambda) \le \int_{\omega+\epsilon}^\infty \lambda^{-2k}\|\lambda^k a(\lambda)\|^2_{A(\lambda)}d\mu \le \|a\|^2_A\,(\omega/\omega+\epsilon)^{2k}$$

which shows that either $a(\lambda)$ is zero on σ or σ has measure zero. The implications $b) \to d) \to c)$ in Theorem 1.1 are evident. So it is enough to show the implication $c) \to b)$ which is a consequence of the following lemma.

Lemma 5.2. Let D be a generator of a one-parameter group of operators e^{tD} in a Banach space B and $\|e^{tD}f\| = \|f\|$. If for some $f \in B$ there exists an $\omega > 0$ such that the quantity

$$\sup_{k\in N}\|D^k f\|\omega^{-k} = R(f)$$

is finite, then $R(f) \le \|f\|$.

Proof. By assumption $\|D^r f\| \le R(f)\omega^r, r \in N$. Now for any complex number z we have

$$\|e^{zD}g\| = \|\sum_0^\infty (z^r D^r g)/r!\| \le R(f)\sum_0^\infty |z|^r\omega^r/r! = R(f)e^{|z|\omega}.$$

It implies that for any functional $h \in B^*$ the scalar function $(e^{zD}f, h)$ is an entire function of exponential type ω which is bounded on the real axis R^1 by the constant $\|h\|\|f\|$. An application of the Bernstein inequality gives

$$\|(e^{tD}D^k f, h)\|_{C(R^1)} = \|\left(\frac{d}{dt}\right)^k (e^{tD}f, h)\|_{C(R^1)} \le \omega^k\|h\|\|f\|.$$

The last one gives for $t = 0$

$$|(D^k f, h)| \le \omega^k\|h\|\|f\|.$$

Choosing h such that $\|h\| = 1$ and $(D^k f, h) = \|D^k f\|$ we obtain the inequality $\|D^k f\| \le \omega^k\|f\|, k \in N$, which gives

$$R(f) = \sup_{k\in N}(\omega^{-k}\|D^k f\|) \le \|f\|.$$

Lemma 5.2 is proved. □

Finally, combining the Bernstein inequality for SE-functions with approximation Theorem 1.5 we are coming to the uniqueness and reconstruction Theorem 1.6 for the SE-functions.

Using Lemma 3.4 and Bernstein inequality one can also show that for SE-functions f the $L^2(M)$ norm is equivalent to the norm $(\sum_{x_\gamma\in Z}|f(x_\gamma|^2)^{1/2}, Z \in Y(r(M),\lambda), \lambda < \lambda_0$.

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